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THE FOURIER TRANSFORM WITH HENSTOCK–KURZWEIL AND CONTINUOUS PRIMITIVE INTEGRALS

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Abstract. For each $f:\mathbb{R}\to\mathbb{C}$ that is Henstock–Kurzweil integrable on the real line, or is a distribution in the completion of the space of Henstock–Kurzweil integrable functions in the Alexiewicz norm, it is shown that the Fourier transform is the second distributional derivative of a Hölder continuous function. The space of such Fourier transforms is isometrically isomorphic to the completion of the Henstock–Kurzweil integrable functions. There is an exchange theorem, inversion in norm and convolution results. Sufficient conditions are given for an L^1 function to have a Fourier transform that is of bounded variation. Pointwise inversion of the Fourier transform is proved for functions in L^p spaces for $1 < p < \infty$. The exchange theorem is used to evaluate an integral that does not appear in published tables.

1. Introduction

In this paper differentiation and integration in spaces of tempered distributions are used to define the Fourier transform of $f:\mathbb{R}\to\mathbb{C}$ when f is Henstock–Kurzweil integrable. All of the results also apply when f is in the completion of the space of Henstock–Kurzweil integrable functions in the Alexiewicz norm, which consists of the distributions that are the distributional derivative of a function that is continuous on the extended real line.

If $f\in L^1(\mathbb{R})$ then its Fourier transform is $\hat{f}(s)=\int_{-\infty}^{\infty}e^{-ist}f(t)\,dt$ for $s\in\mathbb{R}$. In this case, $\hat{f}$ is uniformly continuous on $\mathbb{R}$ and vanishes at infinity (Riemann–Lebesgue lemma). See, for example, [9], [13] or [23] for basic results in Fourier analysis. The dual space and set of multipliers of $L^1(\mathbb{R})$ is $L^\infty(\mathbb{R})$ so the Fourier transform is well defined and $\left|\hat{f}(s)\right|\le\sup_{t\in\mathbb{R}}\left|e^{-ist}\right|\int_{-\infty}^{\infty}|f(t)|\,dt=\|f\|_1$. But when the integral $\int_{-\infty}^{\infty}f(t)\,dt$ is allowed to converge conditionally then the set of multipliers are the functions of bounded

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