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THE FOURIER-PLANCHEREL TRANSFORM DEFINED THROUGH A DENSE SUBSPACE IN $L^2(\mathbb{R})$ WHICH IS NOT CONTAINED IN $L^1(\mathbb{R})$

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Abstract. In this article we present a construction of the Fourier transform over $L^2(\mathbb{R})$. The starting point for this construction is the dense subspace of $L^2(\mathbb{R})$ of the integrable Henstock-Kurzweil functions which are also of bounded variation. This subspace has no containment relations with the space of integrable Lebesgue functions.

1. Introduction

The Fourier transform has been widely studied on the space of integrable functions in the Lebesgue sense. So, if $f \in L^1(\mathbb{R})$, its Fourier transform is defined, for every $s \in \mathbb{R}$, as

$$\mathcal{F}(f)(s) = \widehat{f}(s) = \int_{\mathbb{R}} f(x)e^{-isx}dx. \tag{1.1}$$

In general, when extending this transform to other function spaces, the expression in (1.1) can no longer be defined for every real number or can no longer be represented as (1.1). For example, this happens in the $L^p(\mathbb{R})$ spaces, with $p \in (1, 2]$.

The classical way of constructing the Fourier Transform over $L^2(\mathbb{R})$ is done from the Fourier Transform over $L^1(\mathbb{R})$ and a dense subspace in $L^2(\mathbb{R})$ that is also contained in $L^1(\mathbb{R})$; more precisely, this subspace is chosen with the above conditions such that the Fourier transform operator is a linear isometry on this subspace. Then by a standard extension process this operator is extended to all $L^2(\mathbb{R})$. The classical subspaces used are, among others, $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and the Schwartz space $S(\mathbb{R})$ [3], [5], [13].

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